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PSEUDODIFFERENTIAL OPERATORS WITH ANALYTIC SYMBOLS AND THEIR APPLICATIONS

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This paper contains an exposition of the ideas on the use of infinite order differential operators which are local representatives of pseudodifferential operators with analytic symbols to provide a convenient method for investigating partial differential equations and their initial and boundary value problems.

I. Introduction. The theory of pseudodifferential operators has extensive and fruitful applications to partial differential equations (PDEs) and, among their number, to mathematical physics (see, for example, [1-4]).

In [5] Dubinskii Yu.A. presented a new concept of psuedo-differential operators with constant symbols analytic in an arbitrary domain $G \subset \mathbb{R}^N$ and gave various applications to mathematical physics. The basis of these applications is the nonformal algebra of differential operators of infinite order (DOIO) as the operators acting in the corresponding Sobolev spaces of infinite order.

To illustrate the idea on the use of pseudodifferential operators with analytic symbols (PDOAS) we consider the Dirichlet problem for Laplace equation

$$(1.1) \qquad \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0 \ , \ t > 0 \ , \ x \in \mathbb{R}^1$$

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$$(1.2)\qquad \begin{aligned} &u(0,x) = \varphi(x) \quad , \\ &|u(t,x)| \leq M < +\infty \quad (t \to +\infty) \quad , \end{aligned}$$

where $\varphi(x) \in L_2(\mathbb{R}^1)$.

For solving this problem we put $D \leftrightarrow \frac{1}{i}\partial/\partial x$ considering D as a real parameter and find the solution of ordinary differential equation

$$\frac{d^2U}{dt^2} - D^2U = 0 \ , \ U(0,D) = 1 \ , \ |U(t,D)| \leq M \ .$$

It is easy to see, that $U(t,D) = \exp(-t\,|D|)$ and, consequently, the formal solution of problem (1.1)-(1.2) is presented by the formula

$$(1.3)\qquad u(t,x) = U(t,D)\varphi(x) = \exp(-t\,|D|)\varphi(x) \ ,$$

where D now is the operator of differentiation, i.e. $D = \frac{1}{i}\partial/\partial x$.

The operator $\exp(-t\,|D|)$ has got the symbol $\exp(-t\,|\xi|)$, which has a singularity at $\xi = 0$. At that time, the formula (1.3) has meaning if we put

$$|D|\,\varphi(x) = D\varphi_+(x) - D\varphi_-(x) \quad ,$$

where

$$(1.4)\qquad \varphi_\pm(x) = \frac{1}{2\pi}\int_{\mathbb{R}^1_\pm} \tilde{\varphi}(\xi)\, e^{ix\xi}\, d\xi \ .$$

(Here $\tilde{\varphi}(\xi)$ is denoted the Fourier transformation of $\varphi(x)$). Then the operators $\exp(-t\,|D|)$ acts on $\varphi(x)$ by the formula

$$\exp(-t|D|)\varphi(x) = \exp(-tD)\varphi_+(x) + \exp(tD)\varphi_-(x) \quad ,$$

that is

$$\exp(-t\,|D|)\varphi(x) = \varphi_+(x+it) + \varphi_-(x-it) \quad .$$

By the same token the solution of (1.1)-(1.2) is written in the form

$$u(t,x) = \varphi_+(x+it) + \varphi_-(x-it) \quad ,$$

where the functions $\varphi_\pm$ are defined by (1.4). Substituting these expressions (1.4) into the obtained formula we have got the classical Poisson's integral

$$u(t,x) = \frac{t}{\pi}\int_{\mathbb{R}^1} \frac{\varphi(y)dy}{t^2+(x-y)^2} \quad .$$

Thus, the solvability of (1.1)-(1.2) is established.

Note that the domain of analyticity of the symbol $\exp(-t\,|\xi|)$ consists of two components $\mathbb{R}^1_+$ and $\mathbb{R}^1_-$, which correspond two function spaces, where the operator $\exp(-t|D|)$ acts invariantly as an operator of translation on $\pm$ it, that is as a differential operator of infinite order. Thus, in local the operator $\exp(-t|D|)$ is a differential operator of infinite order acting in corresponding suitable function space and, in this case, DOIO is an instrument of investigation.

The basis of our consideration is the nonformal algebra of DOIO, as the operators acting in the corresponding Sobolev spaces of infinite order. This makes it possible, by considering $\partial/\partial x$ as a parameter in the PDE

$$(1.5) \qquad L(\partial/\partial t\,,\partial/\partial x)u(x,t) = f(x,t) \quad ,$$

to solve (1.5) as an ordinary differential equation, to which are adjoint the initial or boundary conditions.

Considering a PDE in the Sobolev space of infinite order we do not require that this equation has any definite type. The type of the equation has no role. We emphasize, however, that in the process of solving of problems that are well-posed in the spaces of finite smoothness, the Sobolev spaces of infinite order play an intermediate role, being merely an instrument of the investigation. But in the solution of problems, that are ill-posed in the sense of Hadamard-Petrowskii, the introduction of Sobolev spaces of infinite order constitutes the very essence of the approach. It means that the problems which are noncorrect in the classical sense are correct in these spaces.

To illustrate this method we consider the Cauchy problem for the heat inverse equation

$$(1.6) \quad \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0 \ , \ u(0,x) = \varphi(x) \ , \ t > 0 \ , \ x \in \mathbb{R}^1$$

Putting $p = \partial^2/\partial x^2$, we can find that

$$u(t,x) = e^{-t\,\partial^2/\partial x^2}\,\varphi(x) \ .$$

For any $t > 0$ the operator $\exp(-t\,\partial^2/\partial x^2)$ is a ΨDO, the symbol of which is $a(t,\xi) = \exp(t\,\xi^2)$, $\xi \in \mathbb{R}^1$. According to Theorem 4.1 in [5] and Theorem 2.3 below the operator $\exp(-t\,\partial^2/\partial x^2)$ gives isomorphisms

$$\exp(-t\,\partial^2/\partial x^2) : H_a^{\infty} \to L_2(\mathbb{R}^1) \ ,$$

$$\exp(-t\,\partial^2/\partial x^2) : W^{\pm\infty}(\mathbb{R}^1) \to W^{\pm\infty}(\mathbb{R}^1) \ ,$$

where

$$H_a^{\infty} = \left\{ \varphi(x) \in L_2(\mathbb{R}^1) \; , \; \| e^{t\xi^2} \tilde{\varphi}(\xi) \|_{L_2(\mathbb{R}^1)} < +\infty \, , \; t > 0 \right\},$$

$$W^{+\infty} = \left\{ \varphi(x) \in \mathcal{S}' , \ \text{supp } \tilde{\varphi}(\xi) \ \text{ is compact} \right\} , \ W^{-\infty} = (W^{+\infty})' .$$

We remark that, for any initial function $\varphi(x) \in H_a^{\infty} (\varphi(x) \in W^{\pm\infty})$ there exists one and only one solution of the Cauchy problem of the equation (1.6) in the sense of $L_2(\mathbb{R}^1)$ $(W^{\pm\infty})$. After some simple calculations one can get that for any $\varphi(x) \in H_a^{\infty}$ (or $\varphi(x) \in W^{+\infty}$) the solution of problem (1.6) is given by the formula

$$u(t,x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{\frac{(i\xi)^2}{4t}} \varphi(x+i\xi) d\xi \ , \ t > 0 \ .$$

Thus, the technique of ΨDOAS and the introduction of Sobolev spaces of infinite order are the core of described above method : problems which are noncorrect in the classical sense are correct in these spaces.

In this paper we shall give a survey of our results concerning the ΨDOAS, their applications to mathematical physics and approximate methods of solving initial and boundary problems based on the technique of DOIO. In the next section we investigate the test function space $W^{+\infty}$ and the space of generalized functions $W^{-\infty}$ and the properties of DOIOs. In the sections III, IV we give some applications to Cauchy problems and boundary value problems. The last section is devoted to describe the approximation technique based on the use of ΨDOAS and function spaces of infinite order.

II. The ΨDOAS and the spaces of test functions and generalized functions $W^{+\infty}(G)$ and $W^{-\infty}(G)$.

In this section we shall state some our and Trinh Ngoc Minh's results on ΨDOAS. Namely, we construct the function spaces $W^{\pm\infty}$ with the property : The ΨDOAS act invariantly in $W^{\pm\infty}$ and constitute an algebra isomorphic to the algebra of analytic functions in G .

2.1. The space in a neighbourhood of zero. Let $R = R_1,\ldots,R_N$, $0 < R_j \leq +\infty$, $j=1,\ldots,N$, be a vector, and let $S_R = \{ \xi \in \mathbb{R}^N , |\xi_j| < R_j , j=1,\ldots,N \}$ be a parallelepiped. We denote by $W^{+\infty}(S_R)$ the set of function $f(x):\mathbb{R}^N_x \to \mathbb{C}^1$ satisfying the following conditions :

i) f admits analytic continuation as an entire function to $\mathbb{C}^N$.

ii) There exist constants c,m and a vector r such that $r < R$ (i.e., $r_j < R_j$, $j=1,\ldots,N$) and

$$|f(z)| \leq c(1+|z|)^m \exp\Big(\sum_{j=1}^{N} r_j |\mathrm{Im}\, z_j|\Big) .$$

From the Paley-Wiener theorem it follows that $f(x)$ belongs to $W^{+\infty}(S_R)$ if and only if $f \in S'$ and there exists $r < R$ such that $\mathrm{supp}\, \tilde{f}(\xi) \subset S_r$, where $\tilde{f}$ denotes the Fourier transform of f . Examples of functions in $W^{+\infty}(S_R)$ are : all functions in $H^{\infty}(S_R)$ [5] ; all functions in $\mathcal{H}_{yp}$, $1 \leq p \leq +\infty$, $0 \leq y_j \leq R_j$, $j=1,\ldots,N$ [6] ; the quasipolynomials $\exp(i\lambda x).P_m(x)$, $\lambda \in S_R$, and, in particular, the polynomials $1,x,x^2,\ldots$; the trigonometric functions $\prod_1^N \sin x_i$ and $\prod_1^N \cos x_i$; etc.

We introduce a topology in $\overset{+\infty}{W}(S_R)$ as follows. A sequence $f_n(x) \to f(x)$ in $\overset{+\infty}{W}(S_R)$ if

i) $f_n(x) \to f(x)$ uniformly on each compact set $K \subset S_R$, and

ii) there exist constants c, m, r R such that

$$\left| f_n(z) \right| \leq c(1+|z|)^m \exp(r.|\mathrm{Im}\, z|) , \quad n=1,2,\ldots$$

This topology possesses the properties important for us. For example, the space $\overset{+\infty}{W}(S_R)$ is closed ; the imbeddings $H^{\infty}(S_R) \subset \overset{+\infty}{W}(S_R)$ and $\mathcal{M}_{yp} \subset \overset{+\infty}{W}(S_R)$, $0 \leq y \leq R$, $1 \leq p \leq +\infty$, are continuous ; and the Fourier transformation maps $\overset{+\infty}{W}(S_R)$ continuously into $\mathcal{E}'(S_R)$, where $\mathcal{E}'(S_R) = \{ \psi(\xi) \in S'$, $\mathrm{supp}\, \psi \subset S_R \}$ with the convergence $\psi_n(\xi) \xrightarrow{\mathcal{E}'} \psi(\xi)$ if and only if 1) $\psi_n(\xi) \to \psi(\xi)$ in S' and 2) there exists a compact set $K \subset S_R$ such that $\mathrm{supp}\, \psi_n \subset K$, $n=1,2,\ldots$

We now proceed to consider the action of DOIO in $\overset{+\infty}{W}(S_R)$. Let

$$A(D) = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha , \quad a_\alpha \in \mathbb{C}^1 ;$$

$A(D)$ is a differential operator whose symbol

$$A(\xi) = \sum_{|\alpha|=0}^{\infty} a_\alpha \xi^\alpha$$

is an analytic function in S_R .

<u>THEOREM 2.1.</u> A DOIO with a symbol analytic in S_R acts invariantly and continuously in $\overset{+\infty}{W}(S_R)$.

We note that the technique of differential operators of infinite order and fine properties of generalized functions in S' with compact support (these properties are established in

[7]) are used in the proof of Theorem 2.1.

We further denote by $W^{-\infty}(S_R)$ the set of all continuous linear functionals on $W^{+\infty}(S_R)$. In $W^{-\infty}(S_R)$ we introduce the Fourier transformation by the formula

$$\langle \tilde{h}(\xi), \tilde{\varphi}(-\xi) \rangle \stackrel{\text{def}}{=} (2\pi)^N \langle h(x), \varphi(x) \rangle ,$$

where $h \in W^{-\infty}(S_R)$, $\varphi \in W^{+\infty}(S_R)$ and $\tilde{\varphi}(-\xi)$ $\mathcal{E}'(S_R)$; here $\tilde{\varphi}(-\xi)$ is a generalized function acting according to the formula

$$\langle \tilde{\varphi}(-\xi), \psi(\xi) \rangle \stackrel{\text{def}}{=} \langle \tilde{\varphi}(\xi), \psi(-\xi) \rangle \ , \ \psi(\xi) \in C^{\infty}(R^N_{\xi}) .$$

A remarkable property of functions in $W^{-\infty}(S_R)$ is the following : $F(W^{-\infty}(S_R)) \equiv C^{\infty}(S_R)$, where F denotes the Fourier transformation.

<u>THEOREM 2.2.</u> A DOIO with symbol analytic in S_R acts invariantly and continuously in $W^{-\infty}(S_R)$.

2.2. <u>Spaces in an arbitrary domain.</u> Let $G \subset \mathbb{R}^N$ be some domain. We set (1)

$$W^{+\infty}(G) \equiv \left\{ f(x) : f(x) \in S' \ , \ \operatorname{supp} \tilde{f}(\xi) \subset G \right\} .$$

Convergence in $W^{+\infty}(G)$ is defined as follows. A sequence $f_n(x) \to f(x)$ in $W^{+\infty}(G)$ if the following conditions are satisfied

i) $f_n(x) \to f(x)$ uniformly on each compact set $K \subset \mathbb{R}^N_x$.

ii) There exist constants c and m such that $|f_n(x)| \leq$

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(1) In the case $\operatorname{meas} G = \infty$ in the definition of $W^{+\infty}(G)$ it is further required that there exist a compact set $K \subset G$ such that $\operatorname{supp} \tilde{f}(\xi) \subset K$.

$c(1+|x|)^m$, $n=1,2,\ldots$

iii) There exists a compact set $L \subset G$ such that $\operatorname{supp} \tilde{f}_n(\xi) \subset L$, $n=1,2,\ldots$

It is not hard to show that any function $u(x) \in W^{+\infty}(G)$ can be represented in the form

$$u(x) = \sum_{k \in I} u_{\lambda_k}(x) \quad ,$$

where $u_{\lambda_k}(x) \in W^{+\infty}(S_{R^k}(\lambda_k)) \equiv e^{i\lambda_k x} W^{+\infty}(S_{R^k})$, $G \supset S_{R^k}(\lambda_k)$ is a parallelepiped with center at λ_k , and I is a finite set of indices.

Further, let $A(\xi)$ be an arbitrary complex-valued function which is analytic in G . It is then possible to choose $S_{R^i}(\lambda_i)$ so that in each $S_{R^i}(\lambda_i)$ the function $A(\xi)$ can be expanded in the Taylor series

$$A(\xi) = \sum_{|\alpha|=0}^{\infty} a_\alpha(\lambda_i)(\xi-\lambda_i)^\alpha \;, \quad i \in I \;, \; \xi \in S_{R^i}(\lambda_i) \;.$$

For any function $u(x) \in W^{+\infty}(G)$ we define the action of the ΨDO $A(D)$ on $u(x)$ by the formula

$$A(D)u(x) \overset{\text{def}}{=} \sum_{i \in I} \sum_{|\alpha|=0} a_\alpha(\lambda_i)(D-\lambda_i I)^\alpha u_{\lambda_i}(x) \quad ,$$

and by Theorem 2.1 $A(D)u(x)$ is a function in $W^{+\infty}(G)$. Moreover, it can be shown that this definition does not depend on the number of representing the function $u(x)$, i.e., the action is well-defined.

We now denote by $(W^{+\infty}(G))^*$ the space of continuous linear functionals on $W^{+\infty}(G)$, and we set $W^{-\infty}(G)=(W^{+\infty}(-G))^*$.

Let $A(\xi)$ be a function analytic in G. We assign to it a ΨDO $A(D)$ acting in $W^{-\infty}(G)$ according to the formula

$$\langle A(D)h(x),\varphi(x)\rangle \overset{\mathrm{def}}{=} \langle h(x),A(-D)\varphi(x)\rangle , \quad h \in W^{-\infty}(G), \ \varphi \in \overset{+\infty}{W}(G).$$

<u>THEOREM 2.3.</u> A ΨDO $A(D)$ with symbol analytic in G acts invariantly and continuously in $W^{+\infty}(G)$ (and hence also in $W^{-\infty}(G)$). Moreover, the set of all such ΨDO constitutes an algebra isomorphic to the algebra of functions analytic in G.

III. <u>Cauchy problems for systems of PDEs with a distinguished variable</u>. We consider of the Cauchy problem for any system of PDEs of the form

$$(3.1) \quad \frac{\partial^{m_j} u_j(t,x)}{\partial t^{m_j}} + \underset{k=1}{} A_{jk}(t,\frac{\partial}{\partial t},D)u_k(t,x) = h_j(t,x) ,$$

$$(3.2) \quad \partial^k u_j(0,x)/\partial t^k = \varphi_{kj}(x) , \quad k=0,1,\ldots,m_j-1 , \quad j=1,\ldots,\ell,$$

where

$$A_{jk}(t,\partial/\partial t,D) = \sum_{i=0}^{m_j-1} A_{ijk}(t,D)\frac{\partial^i}{\partial t^i} , \quad D = (-i\frac{\partial}{\partial x_1},\ldots,-i\frac{\partial}{\partial x_1})$$

the $m_j > 0$ are integers, the $A_{ijk}(t,D)$ are arbitrary ΨDO and for each t the symbols $A_{ijk}(t,\xi)$ are analytic functions of ξ in some domain $G \subset \mathbb{R}^N$ which depend continuously on $t \in \mathbb{R}^1$.

We establish well-defined solvability of problem (3.1)-(3.2) in the spaces $\overset{+\infty}{W}(G)$ and particularly, within the framework of the theory of generalized function $(W^{+\infty}(G),W^{-\infty}(G))$ any partial differential operator with constant coefficients also has a fundamental solution of the Cauchy problem.

We note that recently in a number of papers (see, for example, [8]), solvability of the Cauchy problem for equations of Kovalevskaya type in the class of all analytic functions (or analytic functionals) has been established. However, it can be shown that there exists an equation of the form (3.1) for which the Cauchy problem cannot be solved in the class of all analytic functions. Therefore, the introduction of a more restricted space (such as $W^{+\infty}$ in this paper) is essential in order to establish well-defined solvability of the Cauchy problem for any differential equation with a distinguished variable t.

For simplicity of the exposition we set $m_j = m$, $j=1,\dots,$. We denote by $W^{\pm\infty,\ell}(G)$ the space of vector-valued functions $u(x)=(u_1(x),\dots,u_\ell(x))$, $u_i(x) \in W^{\pm\infty}(G)$, $i=1,\dots,$, and we denote by $C^k(\mathbb{R}^1, W^{\pm\infty,\ell}(G))$ the space of vector-valued functions $u(t,x)$ which for each $t \in \mathbb{R}^1$ are vector-valued functions in $W^{\pm\infty}(G)$ depending continuously on t together with their derivatives through order k.

THEOREM 3.1. Let $\varphi_k \in W^{+\infty}(G)$ and $h(t,x) \in C^0(\mathbb{R}^1, W^{+\infty,\ell}(G))$ be arbitrary functions. Then there exists a unique solution of the Cauchy problem (1), (2) in the space $C^m(\mathbb{R}^1, W^{+\infty,\ell}(G))$.

THEOREM 3.2. Let $\varphi_k \in W^{-\infty}(G)$ and $h(t,x) \in C^0(\mathbb{R}^1, W^{-\infty,\ell}(G))$ be arbitrary functions. Then there exists a unique solution of the Cauchy problem (1), (2) in the space $C^m(\mathbb{R}^1, W^{-\infty,\ell}(G))$.

The idea of the proof of Theorem 3.1 and 3.2 is as follows. First of all, we observe that by Duhamel's principle it suffices to consider the case $h(t,x) \equiv 0$. Further, the required assertions for a Cauchy problem of first order are proved by a method analogous to that of [5] except for the uniqueness of a solution of the Cauchy problem in the space $W^{+\infty}(G)$ of test functions. In the present situation there arises a difficulty connected with the fact that the Fourier transform of a function in $W^{+\infty}(G)$

can be a genuine generalized function (for example, a delta function). To prove uniqueness of the Cauchy problem in $W^{+\infty}(G)$, by means of a change of variables (see [9] or [8]) we reduce problem (1), (2) to a Cauchy problem for a system of first order and use a method analogous to the familiar Holmgren method.

From Theorem 3.2 it follows, in particular, that for any operator

$$L\left(\frac{\partial}{\partial t}, D\right) \equiv \frac{\partial^m u}{\partial t^m} + \sum_{k=0}^{m-1} A_k(t,D)\frac{\partial^k u}{\partial t^k}$$

a fundamental solution of the Cauchy problem exists and is unique. It is a generalized function in $W^{-\infty}(G)$.

IV. Some concrete examples

In this section we show that the previous results can be applied to some problems of mathematical physics.*)

1. Cauchy problem for one differential equation of relativistic quantum mechanics. In the halfplane R we consider the problem (see J. Bjorken, S. Drell [10], Ch.I, § 1)

$$(4.1) \quad i\frac{\partial u}{\partial t} - \frac{c}{\omega}\sqrt{I-\omega^2\Delta}\,u = 0\ ,\ t>0\ ,\ x\in \mathbb{R}^3\ ,$$

$$(4.2) \quad u(0,x) = \varphi(x)\ ,\ x\in \mathbb{R}^3\ ,$$

where $\omega = \hbar/mc$ (the notations are standard). We shall consider the operator $A(D) = \sqrt{I-\omega^2\Delta}$ as a ΨDO with symbol $A(\xi) = 1+\omega^2\xi^2$. Obviously, $A(\xi)$ is really analytic in full Euclidean space $\mathbb{R}^3$. Consequently, we can apply the results

*) For the data from $H^{\infty}(G)$ the examples 1-3 are considered in [5].

of Section III : for any initial functions $\varphi(x) \in W^{+\infty}(\mathbb{R}^3)$ there exists one unique solution of the problem (4.1)-(4.2)

$$u(t,x) = \exp\left\{\frac{ct}{i\omega}\sqrt{I-\omega^2\Delta}\right\}\varphi(x) \quad .$$

2. Cauchy problem for the Laplace equation. In the half-plane $\mathbb{R}^2_+ = \left\{ t > 0 \ , \ x \in \mathbb{R}^1 \right\}$ we consider the problem

$$(4.3) \qquad \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0 \quad ,$$

$$(4.4) \qquad u(0,x) = \varphi(x) \ , \quad \frac{\partial u}{\partial t}(0,x) = \psi(x) \quad ,$$

where the functions $\varphi(x)$ and $\Psi(x)$ are given. Putting $p = \partial/\partial x$ we have from (4.3)

$$\frac{\partial^2 u}{\partial t^2} + p^2 u = 0 \quad .$$

From this we obtain the formula

$$u(t,x) = e^{itp}c_1(x) + e^{-itp}c_2(x) \quad ,$$

where the functions $c_1(x)$ and $c_2(x)$ are arbitrary. In order to determine these functions we use the initial conditions (4.4), from which

$$c_1(x) + c_2(x) = \varphi(x) \quad ,$$

$$ip\, c_1(x) - ip\, c_2(x) = \Psi(x) \quad .$$

After elementary calculations we get the formula

$$u(t,x) = \frac{e^{itp} + e^{-itp}}{2}\varphi(x) + \frac{e^{itp} - e^{-itp}}{2ip}\psi(x) \ .$$

Taking into account the relation $p = \partial/\partial x$, we find that the desired solution has the form

(4.5) $$u(t,x)= \frac{e^{it\,\partial/\partial x}+e^{-it\,\partial/\partial x}}{2}\varphi(x)+ \frac{e^{it\partial/\partial x}-e^{-it\partial/\partial x}}{2i\partial/\partial x}\,\Psi(x) ,$$

or, equivalently,

(4.6) $$u(t,x) = [\cos t\partial/\partial x]\varphi(x) + [\frac{\sin t\partial/\partial x}{i\partial/\partial x}]\,\Psi(x) .$$

Let us elucidate what the initial functions $\varphi(x)$ and $\Psi(x)$ should be in order that the formulas (4.5) and (4.6) have a nonformal sense. It is obvious that the operator

$$e^{it\partial/\partial x} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\frac{\partial^n}{\partial x^n}$$

acts invariantly and continuously in $W^{+\infty}(\mathbb{R}^1)$, moreover, for any $\varphi(x)$:

$$e^{it\partial/\partial x}\varphi(x) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\frac{\partial^n}{\partial x^n}\varphi(x) = \varphi(x+it) ,$$

since any function $\varphi(x)$ from the space $W^{+\infty}(\mathbb{R}^1)$ is an entire function. It follows, consequently, that the first term in (4.5) and (4.6) has a nonformal sense when $\varphi(x) \in W^{+\infty}(\mathbb{R}^1)$

$$\frac{e^{it\partial/\partial x} + e^{-it\partial/\partial x}}{2}\varphi(x) = \frac{\varphi(x+it) + \varphi(x-it)}{2} .$$

Putting

$$\left(\frac{\partial}{\partial x}\right)^{-1}\Psi(x) = \int_0^x \Psi(\xi)\,d\xi + c ,$$

where c is an arbitrary constant, we get that the second term in (4.5), (4.6) is also determined for any function $\Psi(x) \in W^{+\infty}(\mathbb{R}^1)$; moreover,

$$\frac{e^{it\partial/\partial x} - e^{it\partial/\partial x}}{2i\partial/\partial x}\psi(x) = \frac{1}{2i}\int_{x-it}^{x+it}\psi(\xi)\,d\xi \;.$$

Thus, for any $\varphi \in W^{+\infty}(\mathbb{R}^1)$, $\psi \in W^{+\infty}(\mathbb{R}^1)$ the formulas (4.5) and (4.6) give the formula of d'Alembert's type

$$u(t,x) = \frac{\varphi(x+it) + \varphi(x-it)}{2} + \frac{1}{2i}\int_{x-it}^{x+it}\psi(\xi)\,d\xi \;.$$

Finally, we note that the fundamental solution of the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0 \;,\; u(0,x) = 0 \;,\; \frac{\partial u}{\partial t}(0,x) = \delta(x) \quad ,$$

has got the form

$$\mathcal{E}(t,x) = \left[\frac{\sin t\partial/\partial x}{i\partial/\partial x}\right]\delta(x) \;.$$

As in [5, §2] one can obtain that on test function $\varphi(x) \in W^{+\infty}(\mathbb{R}^1)$,

$$\mathcal{E}(x,t) = \frac{1}{2i}\left[\theta(x+it) - \theta(x-it)\right] \;,$$

where $\theta(x \pm it)$ is a complex translation of Heaviside function.

3. Boundary value problem in the strip. Let $G = \{ -1/2 < t < 1/2 \;,\; x \in \mathbb{R}^1\}$ be the strip of the variables (t,x). We consider the following boundary value problem

$$(4.7) \quad \frac{\partial^2 u}{\partial t^2} + a^2\frac{\partial^2 u}{\partial x^2} = 0 \;,\quad a \in \mathbb{C}^1 \;,\quad |a| = 1 \;,$$

$$(4.8) \quad u(1/2,x) = \varphi_+(x) \;,\; \frac{\partial u}{\partial t}(-1/2,x) = \varphi_-(x) \quad .$$

Putting $\frac{1}{i}\partial/\partial x \equiv D \leftrightarrow \xi$ and consider two boundary value problem for ordinary differential equation with the parameter ξ :

$$u''_{\pm}(t,\xi) - a^2\xi^2 u_{\pm}(t,\xi) = 0$$

under the boundary conditions

$$u_+(1/2,\xi) = 1 \ , \quad u'_+(-1/2,\xi) = 0 \ ;$$

$$u_-(1/2,\xi) = 0 \ , \quad u'_-(-1/2,\xi) = 1 \ .$$

The direct calculation shows that

$$u_+(t,\xi) = \frac{\operatorname{cha}(t+1/2)\xi}{\operatorname{cha}\xi} \ , \quad u_-(t,\xi) = \frac{\operatorname{sha}(t-1/2)\xi}{a\xi\operatorname{cha}\xi} \ .$$

Consequently, the solution of problem (4.7)-(4.8) accepts a form

$$u(t,x) = U_+(t,D)\varphi_+(x) + U_-(t,D)\varphi_-(x) \equiv$$

$$\equiv \frac{\operatorname{cha}(t+1/2)D}{\operatorname{cha}D}\varphi_+(x) + \frac{\operatorname{sha}(t-1/2)D}{aD\operatorname{cha}D}\varphi_-(x) \ ,$$

or,

$$u(t,x) = \operatorname{cha}(1/2+t)Du_+(x) + \frac{\operatorname{sha}(t-1/2)D}{aD}u_-(x) \ ,$$

where the functions $u_{\pm}(x)$ are solutions of the equations

$$[\operatorname{cha}D]u_{\pm}(x) = \varphi_{\pm}(x) \ .$$

Let $a = \sigma + i\tau$, $\sigma \in \mathbb{R}^1$, $\tau \in \mathbb{R}^1$. The behaviour of the symbol $\operatorname{cha}\xi \equiv \cos ai\xi$ essentially depends on σ . Namely, if $\sigma = 0$ then $\cos ai\xi = \cos\tau\xi = 0$ when $\tau\xi = \pi/2 + k\pi$, $k=0,\pm1,\dots$ In this case the domain of analyticity of the symbol $1/\cos\tau\xi$ is $G = \mathbb{R}^1 \setminus \{\tau^{-1}(\pi/2 + k\pi)\}$. Thus, in the case of wave equation $(a = i)$ the Dirichlet problem has one and only one solution for any $\varphi_{\pm}(x) \in W^{\pm\infty}(G)$. By the classical point of view, this problem is ill-posed.

If $\sigma \neq 0$, then for any σ we have on the ray $z = a\xi$ the relation $|\mathrm{ch} z| \sim \mathrm{ch}\sigma\xi$, and, consequently, the Dirichlet problem is well-posed not only in the space $W^{+\infty}(\mathbb{R}^1)$ but also in the space of finitely smooth functions with weight $\mathrm{ch}\sigma\xi$. In particular, when $\mathrm{Re}a \neq 0$ the Dirichlet problem (4.7)-(4.8) is well-posed in the Sobolev spaces $H^m(\mathbb{R}^1)$, i.e. well-posed in the classical sense.

4. In this and the following examples we consider the case of function spaces with weights. Denote the function space

$$H_\rho^\infty(G) \equiv \Big\{ \varphi(x), \quad \mathrm{supp}\,\tilde{\varphi}(\xi) \subset \bar{G} \ ,$$
$$\|\varphi\|_\rho^2 = \int_G |\tilde{\varphi}(\xi)|^2 \rho^\alpha(\xi) d\xi < +\infty \Big\} ,$$

where $\rho(\xi) = \min\limits_{\tau \in \partial G} \|\xi - \tau\|_{\mathbb{R}^N}$, $\|\xi - \tau\|_{\mathbb{R}^N}$ is the distance from ξ to τ in R^N . We investigate the Helmholtz's equation

$$\Delta u(x) + \omega^2 u(x) = h(x) \quad ,$$

where ω is a real parameter. Let $h(x) \in H_\rho^\infty(G)$ (or $h(x)$ $H_\rho^{-\infty}(G)$) , where $G = \{ \xi \in \mathbb{R}^N , \xi^2 \leq \omega^2 \}$. Then the unique solution $u(x)$ is presented in the form $u(x)=[I/(\Delta+\omega^2 I)]h(x)$.

5. Consider the equation

$$\sin(D)u(x) = h(x) \ , \ h(x) \in H_\rho^\infty(G) \quad ,$$

where $G = [k\pi, (k+1)\pi]$, k is an integer. Then

$$u(x) = (I/\sin(D))h(x) \ .$$

6. Let $G = \{ \xi \in \mathbb{R}^N : \|\xi\|_{\mathbb{R}^N} \leq 1 \}$. Consider the equation

$$\Delta u = h(x) \ , \quad h(x) \in H_{\rho}^{\pm\infty}(G) \ ,$$

where $\rho(\xi) = \|\xi\|_{\mathbb{R}^N}$. Then, for any $h(x) \in H_{\rho}^{\pm\infty}(G)$ there exists an unique solution $u(x) \in H^{\pm\infty}(G)$

$$u(x) = (I/\Delta)h(x) \ .$$

V. Approximation methods

This section contains some our and Dinh Nho Hao's results concerning the approximate methods of solving PDEs based on the technique of DOIO .

The technique of DOIO has been used by many mathematicians and mechanicians in recent years to solve broad classes of ODEs and PDEs (see, for example [11,12]). It is this technique we shall make use of to approximate the image of a function φ in $W^{\pm\infty}(G)$ under a certain operator $A(D)$ with the symbol $A(\xi)$ analytic in a bounded domain $G \subset \mathbb{R}^N$. It is well known that for many ODEs or PDEs the problem of finding their solutions is reduced to thatof calculating the image of a function in $W^{\pm\infty}(G)$ under a certain PDO $A(D)$ (Sections 2,3 above). For example, for the Cauchy problem $\frac{\partial u}{\partial t} = L(D)u$, $u|_{t=0} = \varphi(x)$, the role of $A(D)$ is taken by $e^{tL(D)}$, for the problem $L(D)u = h$, the role of $A(D)$ is taken by $\frac{I}{L(D)}$, etc.

Thus, our problem is reduced to that of finding a sequence of operators $A_n(D)$ with analytic symbols $A_n(\xi)$ such that for a function $\varphi(x)$ given in $W^{\pm\infty}(G)$ the members $A_n(D)\varphi(x)$ can be relatively easily found and the sequence $A_n(D)\varphi(x)$ rapidly converges to $A(D)\varphi$ in a desired norm. For this purpose, we shall make use of algebraic polynomials and trigonometric polynomials to approximate the symbols $A(\xi)$.

Note that our approximation methods are based also on the approximation of smooth functions by entire functions of exponential type. Namely, the following assertions are valid.

Lemma 5.1. Let $f(x)$ belong to Sobolev space $W^m_p(\mathbb{R}^N)$, $1 \leq p \leq +\infty$. Then there exists a entire function g_ν of exponential type ν , $g_\nu \in \mathcal{M}_{\nu p}$ such that the estimates are hold :

a) $\| f - g \|_p \leq \frac{c}{\nu^m} \Omega(f^{(m)}, \frac{1}{\nu})_p$,

b) For $\alpha \in \mathbb{Z}^N_+$, $|\alpha| \leq m$

$$\| D^\alpha f - D^\alpha g \|_p \leq \frac{c}{\nu^{m-|\alpha|}} \sum_{|\beta|=m} \Omega(f^{(\beta)}, \frac{1}{\nu})_p .$$

where $(f, \frac{1}{-})_p$ is a module of continuity, and

$$\Omega(f^{(\beta)}, \frac{1}{\nu})_p \to 0 , \nu \to \infty .$$

Lemma 5.2. Let $f(x) \in C^m(\mathbb{R}^N)$. Then there exists a sequence of entire function g_{ν_k} of exponential type ν_k ($g_{\nu k} \in S\mathcal{M}_{\nu k \infty}(\mathbb{R}^N)$) , such that for any compact $K \subset \mathbb{R}^N$

$$\| f - g_{\nu k} \|_{C^m(K)} \to 0 , k \to \infty .$$

The proof of these lemmas follows from Theorem 5.2.4 [6].

By virtue of Lemmas 5.1, 5.2 and the fact that the space $W^{+\infty}(R^N)$ contains all $\mathcal{M}_{\nu p}$, $1 \leq p \leq \infty$, we can approximate smooth functions or the functions from $W^m_p(\mathbb{R}^n)$ by the functions from $W^{+\infty}(\mathbb{R}^N)$.

Thus, for the initial and boundary value problems, the data of which is functions of finite smoothness, our approximation method consists of two steps :

First step : Approximation of the smooth functions by the functions from $W^{+\infty}(G)$

Second step : Finding approximation solutions in $W^{+\infty}(G)$. We below will discuss the second step in detail .

5.1. Approximation of the symbol $A(\xi)$ by algebraic polynomials

a) Approximation by Taylor series

Let $A(\xi)$ be an analytic function in

$$\Delta_\nu = \{x \mid x \in \mathbb{R}^N , |x_j| < \nu_j , \nu_j > 0 , j \in \{1,2,\dots,N\}\},$$

$$A(\xi) = \sum_{|\alpha| \geq 0} a_\alpha \xi^\alpha , \sum_{|\alpha| \geq 0} |a_\alpha| \nu^\alpha < +\infty ,$$

where $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_N)$, $\nu^\alpha = \nu_1^{\alpha_1} \dots \nu_N^{\alpha_N}$ is standard multi-index notation.

We must approximate the expression

$$g(x) = A(D)\varphi(x) = \sum_{|\alpha| \geq 0} a_\alpha D^\alpha \varphi(x) , \varphi \in W^{+\infty}(\Delta_\nu) .$$

Since $\sum_{|\alpha| \geq 0} |a_\alpha| \nu^\alpha < +\infty$, there exists a vector $\varepsilon = (\varepsilon_1,\varepsilon_2,\dots,\varepsilon_N)$ where $\varepsilon_i > 0$ such that $\sum_{|\alpha| \geq 0} |a_\alpha| (\nu + \varepsilon)^\alpha < +\infty$. The function $A(\xi)$ is analytic in $\Delta_{\nu+\varepsilon}$, and it can be analytically extended into $\square_{\nu+\varepsilon} = \{z, z_j = x_j + iy_j, |x_j| < \nu_j + \varepsilon_j , |y_j| \leq \nu_j + \varepsilon_j , j \in [1,2,\dots,N]\}$.

The following theorem shows that the functions $g_n = A_n(D)\varphi =$

$= \sum_{|\alpha| \leq n} a_\alpha D^\alpha \varphi$ corresponding to the Taylor polynomial $A_n(\xi) =$ $= \sum_{|\alpha| \leq n} a_\alpha \xi^\alpha$ of the function $A(\xi)$ can approximate $A(D)\varphi$.

Theorem 5.3. If $\varphi \in W^{+\infty}(\Delta_\nu)$, then $g(x) = \sum_{|\alpha| \geq 0} a_\alpha D^\alpha \varphi(x)$ converges in the topology of the space $W^{+\infty}(\Delta_\nu)$ (see the sections II and III). Further, if $\varphi \in \mathcal{H}_{\nu p}$ then $g(x) = \sum_{|\alpha| \geq 0} a_\alpha D^\alpha \varphi(x)$ converges in the norm of $L_p(\mathbb{R}^N)$ and the following inequalities hold :

$$\|g\|_p = \|A(D)\varphi\|_p \leq \bar{A}(\nu)\|\varphi\|_p \leq \max_{z \in \partial \square_{\nu+\varepsilon}} |A(z)| \left(1 + \frac{\varepsilon}{\nu}\right) \|\varphi\|_p ,$$

$$\|g\|_2 = \|A(D)\varphi\|_2 \leq \max_{\xi \in \Delta_\nu} |A(\xi)| \, \|\varphi\|_2 ,$$

$$\|g - g_n\|_p \leq \Big(\sum_{|\alpha| > n+1} |a_\alpha| \nu^\alpha \Big) \|\varphi\|_p \leq \max_{z \in \partial \square_{\nu+\varepsilon}} |A(z)| \left(1 + \frac{\varepsilon}{\nu}\right)^n \|\varphi\|_p ,$$

$$\|g - g_n\|_2 \leq \max_{\xi \in \Delta_\nu} \frac{|A^{(n+1)}(\xi)|}{(n+1)!} \|\varphi\|_2 \to 0 \quad \text{as} \quad n \to \infty ,$$

where $\bar{A}(\nu) = \sum_{|\alpha| \geq 0} |a_\alpha| \nu^\alpha$, $1 + \frac{\varepsilon}{\nu} = \left(\left(1 + \frac{\varepsilon_1}{\nu_1}\right) \ldots \left(1 + \frac{\varepsilon_N}{\nu_N}\right)\right)$

The proof of this theorem follows by an estimation of the Taylor coefficients a_α of $A(\xi)$ and the Bernstein-Nikolskii inequalities ([6]).

Now, we consider the following Cauchy problem

$$\frac{\partial u}{\partial t} - A(D)u = 0 ,$$

$$u\big|_{t=0} = \varphi \in W^{+\infty}(\Delta_\nu) .$$

By the Theorem 3.1 we have $u(t,x) = e^{tA(D)}\varphi(x) = \sum_{k=0}^{\infty} \frac{t^k A^k(D)}{k!}\varphi(x)$ and this expression is correctly defined in $W^{+\infty}(\Delta_y)$. The following theorem shows that $u_n(t,x) = \sum_{k=0}^{n} \frac{t^k A^k(D)}{k!}\varphi(x)$ in fact is an approximate solution of the problem.

Theorem 5.4. If $\varphi \in \mathcal{H}_{\gamma p}$ then

$$\| u(t,.)-u_n(t,.) \|_p \leq \frac{[tA(\gamma)]^{n+1}}{(n+1)!} e^{t\bar{A}(\gamma)} \| \varphi \|_p ,$$

$$\|u(t,.)-u_n(t,.)\|_2 \leq \frac{1}{(n+1)!} (t \max_{\xi\in\Delta_y} A(\xi))^{n+1} \exp[t \max_{\xi\in\Delta_y} A(\xi)] \|\varphi\|_2$$

The right hand sides tend to zero as $n \to \infty$.

The Taylor series is a good approximation only in a sufficiently small neighbourhood of the point of expansion. Outside of this neighbourhood it can be a very bad approximation. Consider, e.g., the Cauchy problem for the backward heat equation

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0 ,$$

$$u\Big|_{t=0} = \sin kx \in \mathcal{H}_{k\infty} .$$

We have

$$u(t,x) = e^{-t\frac{d^2}{dx^2}} \sin kx = \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} \frac{d^{2m}}{dx^{2m}} \sin kx$$

$$= \sum_{m=0}^{\infty} \frac{(-t)^m(-k^2)^m}{m!} \sin kx = e^{tk^2} \sin kx .$$

If we take $u_n(t,x) = \sum_{m=o}^{n} \frac{(-t)^m \frac{d^{2m}}{dx^{2m}}}{m!} \sin kx$.

as approximate solution, then for large numbers k this sequence converges very slowly. In fact, for $t = 1$ we have

$$\| u_n(1,x)-u(1,x) \|_{L_\infty(\mathbb{R})} = \| \sum_{m=n+1}^{\infty} \frac{k^{2m}}{m!} \sin kx \|_{L_\infty} = \sum_{m=n+1}^{\infty} \frac{k^{2m}}{m!} .$$

If $k = 10$, then for $n = 100$ the inaccuracy of approximation is larger than $\frac{100^{101}}{101!} > \frac{e^{100}}{30}$. To improve upon this shortcoming of Taylor series we shall use Bernstein polynomials, Lagrange polynomials and the polynomials of best approximation for the symbol $A(\xi)$ in Δ_ν .

b) Approximations by other algebraic polynomials

Let $G = \{ \xi : 0 \leq \xi_i \leq 1 , i \in [1,2,\ldots,N] \}$. We consider the Bernstein polynomials

$$B_n(A,\xi) = \sum_{k_1=o}^{n} \cdots \sum_{k_N=o}^{n} (\frac{k^1}{n},\ldots,\frac{k_n}{n}) p_{nk_1}(\xi_1)\ldots p_{nk_N}(\xi_N)$$

where $p_{nk_j}(\xi_j) = \binom{n}{k_j} \xi_j^{k_j}(1-\xi_j)^{n-k_j}$. If $A' \in \mathrm{Lip}_{M_j} 1$ for every variable ξ_j ([13], p.102) then

$$| A(\xi) - B_n(A,\xi) | \leq \sum_{j=1}^{N} M_j \frac{\xi_j(1-\xi_j)}{2n} .$$

Let $N = 1$, $G = [a,b]$ and $a = \xi_o < \xi_1 < \ldots < \xi_n = b$. Set $\psi_n(\xi) = (\xi - \xi_o)\ldots(\xi - \xi_n)$,

$$L_n(\xi) = \sum_{j=0}^{n} \frac{A(\xi_j)\,\psi_n(\xi)}{(\xi-\xi_j)\,\psi_n'(\xi_j)} \quad .$$

We have

$$R_n(\xi) = |A(\xi)-L_n(\xi)| = \frac{|A^{(n+1)}(t\xi)|}{(n+1)!}|\psi(\xi)| \text{ , where } 0<t<1 \text{ .}$$

If $\psi_n(\xi) = T_n(\xi)$ is the Chebyshev polynomial of degree n ([14], p.14 ff) , then

$$R_n(\xi) < \frac{|A^{(n+1)}(t\xi)|}{(n+1)!} \frac{(b-a)^n}{2^{2n-1}} \text{ .}$$

Let $N = 1$ and $G = [-1,1]$ and $P_n(x)$ be, for every n , the polynomial of best approximation of $A(\xi)$ in G . Set

$$E_n(A) = \inf_{Q\in P_n} \sup_{\xi\in G} |Q(\xi) - A(\xi)| = \sup_{\xi\in G} |P_n(\xi) - A(\xi)| ,$$

where P_n is the space of polynomials of degree at most n . From [13] p.66 we have

$$E_n(A) < M_p \Delta_n^p(\xi)\,\omega(A^{(p)}, \Delta_n(\xi)) \quad ,$$

$$n \in \{p,p+1,\dots\}, \Delta_n = \Delta_n(\xi) = \max\left(\frac{\sqrt{1-\xi^2}}{n}, \frac{1}{n^2}\right) ,$$

$\Delta_o(\xi) = 1$. Here p is an arbitrary natural number.

From these estimates and Theorem 5.3 we can prove the following theorem.

<u>Theorem 5</u>.5. If $\varphi \in H^{\infty}(G)$ ([5]) then

$$\| A(D)\varphi - B_n(A,D)\varphi \|_2 \leq C \frac{1}{n} \|\varphi\|_2 \quad ,$$

$$\| A(D)\varphi - L_n(D)\varphi \|_2 \leq \frac{\max\limits_G |A^{(n+1)}(\xi)|}{(n+1)!} \frac{(b-a)^{n+1}}{2^{2n-1}} \|\varphi\|_2 ,$$

$$\| A(D)\varphi - P_n(D)\varphi \| \leq E_N(A) \|\varphi\|_2 < C \frac{1}{n^2} \|\varphi\|_2 .$$

5.2. Approximation by trigonometric polynomials

Let $\Delta_\nu = \{ x : |x_j| \leq \nu_j \ , j \in \{1,\dots,N\}\}$. From [5] we have $H^\infty(\Delta_\nu) = \bigcup\limits_{\sigma \leq \nu} \mathcal{M}_{\sigma 2}$. If $A(\xi)$ is a periodic function with period 2ν, analytic for all $\xi \in \mathbb{R}^N$, then $A(\xi)$ can be approximated by trigonometric polynomials. If $A(\xi)$ is not 2ν-periodic, then we can extend it as a $2(\nu+\varepsilon)$-periodic function in $\mathbb{R}^N$ with an $\varepsilon > 0$ and it also can be approximated by trigonometric polynomials. In fact, for any function $\psi \in C_0^\infty(\Delta_{\nu+\varepsilon})$, $\psi(\xi) \geq 0$, $\psi(\xi) \equiv 1$ in Δ_ν (such function always exists ([6])) the function $\psi(\xi)A(\xi)$ can be extended to a $2(\nu+\varepsilon)$-periodic function in $C^\infty(\mathbb{R}^N)$, and we have $\psi(\xi)A(\xi) = A(\xi)$ in Δ_ν.

Hence

$$A(\xi) = \sum_{|k| \geq 0} A_k e^{ik\frac{\xi\pi}{\nu+\varepsilon}} ,$$

where $A_k = \dfrac{2^{-N}}{(\nu_1+\varepsilon_1)\dots(\nu_N+\varepsilon_N)} \displaystyle\int_{\Delta_{\nu+\varepsilon}} \psi(\xi)A(\xi)\, e^{ik\frac{\xi\pi}{\nu+\varepsilon}} d\xi$.

Here take $\varepsilon = 0$ if $A(\xi)$ has period 2ν.

Set $A_n(\xi) = \displaystyle\sum_{0 \leq |k| \leq n} A_k\, e^{ik\frac{\xi\pi}{\nu+\varepsilon}}$.

It is clear that

$$A_n(D)\varphi(x) = \sum_{0 \le |k| \le n} A_k \varphi(x + \frac{k\pi}{\gamma+\varepsilon}) \quad ,$$

This formula is very convenient for practical use.

Theorem 5.6. If $\varphi \in \mathcal{M}_{\gamma p}$ then $A_n(D)\varphi(x)$ converges to $A(D)\varphi(x)$ in the $L_p(\mathbb{R}^N)$-norm. Further, let $\mu = (\lambda, \dots, \lambda)$ be a vector whose components are all equal to a natural number λ, and let for any nonnegative integer vector $1 \le \mu$ be $|A^{(1)}(\xi)| \le M$. Then

$$\| A(D)\varphi - A_n(D)\varphi \|_p \le \frac{CM}{n^{\lambda - \frac{1}{2}}} \| u \|_p \quad ,$$

where C depends on λ but does not depend on M and n. If $\Delta_{\gamma+\varepsilon} = [-\pi, \pi]$ then

$$\| A(D)\varphi - A_n(D)\varphi \|_p \le \varepsilon(n) \frac{\log n}{q} \| \varphi \|_p \quad ,$$

where q is any natural number, $\varepsilon(n) \le m_q \omega(A^{(q)}, \frac{1}{n})$, $n \ge 2$.

Proof. We have

$$\| A(D)\varphi - A_n(D)\varphi \|_p \le \sum_{|k| \ge n+1} |a_k| \, \| \varphi \|_p \quad .$$

On the other hand, from [13] $\sum_{|k| \ge n+1} |a_k| \le \frac{CM}{n^{\lambda - \frac{1}{2}}}$, where C depends on λ but does not depend M and n. The first inequality of the theorem is proved.

The second inequality follows from the inequality

$$\sum_{|k| \ge n+1} |a_k| \le \varepsilon(n) \frac{\log n}{q} \; , \; \varepsilon(n) \le m_q \omega(A^{(q)}, \frac{1}{n}).$$

See [15, p 295/296]

Corollary. If $N = 1$ and $G = [-\pi,\pi]$, then there exists a number θ with $0 < \theta < 1$ and a constant C such that

$$\|A(D)\varphi - A_n(D)\varphi\|_p \leq 2C\frac{\theta^{n+1}}{1-\theta}\|\varphi\|_p .$$

In fact, from [16] it follows that there exist θ, $0 < \theta < 1$, and C such that $|A_n| \leq C\theta^n$, $n \in \{0,\pm1,\pm2,\dots\}$. Hence

$$\|A(D)\varphi-A_m(D)\varphi\|_p \leq \sum_{|n|\geq m+1}|A_n|\,\|\varphi\|_p \leq C\sum_{|n|\geq m+1}\theta^{|n|} = 2C\frac{\theta^{m+1}}{1-\theta} .$$

In cases, where it is difficult to calculate the Fourier coefficients of $A(\xi)$ we can use trigonometric interpolation. Let $\Delta_{\nu+\varepsilon} = [-\pi,\pi]$, $N = 1$. Consider $2n+1$ different points $t^n_{-n}, t^n_{-n+1},\dots,t^n_0,\dots,t^n_n \in (-\pi,\pi]$. For $A \in CP_\infty$ ([15], Chapter 5) we take the interpolated trigonometric polynomial $r_n \in V_n$, where V_n is the $(2n+1)$-dimensional space whose basis is $\{1, \cos x, \sin x,\dots, \cos nx, \sin nx\}$. It is easy to verify that r_n can be written in the following form

$$r_n = \sum_{j=-n}^{n} A(t^n_j)m^n_j ,$$

where

$$m^n_j = \prod_{\substack{k\in\{-n,\dots,n\}\\ k\neq j}} \frac{\sin((t-t^n_k)/2)}{\sin((t^n_i-t^n_k)/2)} .$$

In the special case, where $t^n_j - t^n_{j-1} = \frac{2\pi}{2n+1}$ $j \in \{n,\dots,n+1\}$, we have

$$m^n_j = \frac{1}{2n+1}\,\frac{\sin((2n+1)(t-t^n_j)/2)}{\sin((t-t^n_j)/2)}$$

for $t_j^n = t_o^n + jh$, $j = \{-n,..,n\}$, $h = \frac{2\pi}{2n+1}$, $-\frac{\pi}{2n+1} < t_o^n < \frac{\pi}{2n+1}$. Hence, from [15] (Chapter 5) we obtain the estimate

$$\| A(D)\varphi - r_n(D)\varphi \|_p \leq \varepsilon(n) \frac{\log n}{n^q} \| \varphi \|_p ,$$

where q is any natural number, $\varepsilon(n) \leq m_q'(A^{(q)}, \frac{1}{n})$.

5.3. Examples. As remarked above, in order to find solutions of Cauchy problems in spaces $W^{+\infty}$, one must calculate the result of applying a pseudo-differential operator A with analytical symbol to functions $\varphi \in W^{+\infty}$. For this purpose we use algebraic polynomials and trigonometric polynomials to approximate the symbol $A(\xi)$. In this section we demonstrate the idea of trigonometric approximations of the symbol $A(\xi)$. After approximating $A(\xi)$ by trigonometric polynomials we obtain that $A(D)\varphi(x)$ is represented only by translations of $\varphi(x)$. (It is clear that in this case differential operators of infinite order are translation operators). These representations of solutions are new and comfortable for applications.

1) Helmholtz's problem for $G = [-\pi,\pi]$

$$(-\frac{d^2}{dx^2} + a^2)\, u(x) = h(x) .$$

We have $u(x) = \dfrac{I}{-\frac{d^2}{dx^2} + a^2} h(x)$. The symbol of the operator $\dfrac{I}{-\frac{d^2}{dx^2} + a^2}$ is $\dfrac{1}{\xi^2 + a^2}$ and in $[-\pi,\pi]$ we have $\dfrac{1}{\xi^2 + a^2} = \dfrac{1}{2\pi} \sum_{k=-\infty}^{\infty} c_k e^{ik}$, where $c_k = \int_{-\pi}^{\pi} \frac{e^{-ik}}{\xi^2+a^2} d\xi = \frac{i}{2a}[e^{ka}Ei(-ka+ik\pi)-$

$- e^{-ka}Ei(ka+ik\pi) - e^{ka}Ei(-ka-ik\pi) + e^{-ka}Ei(ka-ik\pi)]$ ([16], p.139).

On the special function $Ei(z)$ one can read in [17], Chapter 3, where we can find that $Ei(z) = C + \ln(-z) + \sum_{k=1}^{\infty} \frac{z^k}{k!k}$,

$|\arg(-z)| \angle \pi$, where $C = 0{,}5772157\ldots$ is Euler's constant, $\ln z = \ln|z| + i\arg z$, $|\arg z| \angle \pi$. Thus, we have the following formula of the solution of our problem

$$u(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} c_k\, u(x+k) .$$

2) The Cauchy problem for the heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} , \quad u\Big|_{t=0} = \varphi \in \mathcal{H}_{\nu p} .$$

We have $u(t,x) = e^{ta^2 \frac{\partial^2}{\partial x^2}} \varphi(x)$. The operator $e^{ta^2 \frac{\partial^2}{\partial x^2}}$ has a symbol $e^{-ta^2 \xi^2} = 2\nu \sum_{k=-\infty}^{\infty} d_k(t) e^{i \frac{\pi}{\nu} k\xi}$, where $d_k(t) =$

$$= \int_{-\nu}^{\nu} e^{-ta^2\xi^2 - i\frac{\pi}{\nu}k\xi}\, d\xi = \int_{-\nu}^{\nu} e^{-(a\sqrt{t}\,\xi - \frac{i\pi k}{2\nu a\sqrt{t}})^2 - \frac{\pi^2 k^2}{4a^2 t\nu^2}}\, d\xi =$$

$$= \frac{2}{a\sqrt{t}}\, e^{-\frac{\pi^2k^2}{4a^2t^{2}}}\, \mathrm{erf}\left(a\sqrt{t} + \frac{i\pi k}{2\, a\sqrt{t}}\right)$$, see [16], p.139, and [17].

Notice that $\mathrm{erf}\, z = \int_0^z e^{-s^2} ds$. Thus we have

$$u(t,x) = 2\nu \sum_{k=-\infty}^{\infty} d_k(t)\varphi(x+\frac{k\pi}{\nu}) .$$

3) The Cauchy problem for the inverse heat equation

$$\frac{\partial u}{\partial t} = -a^2\frac{\partial^2 u}{\partial x^2} , \quad u\Big|_{t=0} = \varphi \in \mathcal{H}_{\nu p} .$$

We have $u(t,x) = e^{-a^2 t \frac{\partial^2}{\partial x^2}} \varphi(x)$. The operator $e^{-a^2 t \frac{2}{x^2}}$ has a symbol $e^{a^2 t \xi^2} = 2\nu \sum\limits_{k=-\infty}^{\infty} b_k(t)\, e^{i\frac{\pi}{\nu}k\xi}$, where

$$b_k(t) = \int_{-\nu}^{\nu} e^{a^2 t \xi^2}\, e^{-i\frac{\pi}{\nu}k\xi}\, d\xi = \frac{2}{a\sqrt{t}} \exp\frac{\pi^2 k^2}{4a^2 t \nu^2}\, \mathrm{erfi}\left(a\sqrt{t}\,\nu - \frac{i\pi k}{2\nu a\sqrt{t}}\right),$$

see [16], p.139, [17]. Here $\mathrm{erfi}\, z = \int_0^z e^{s^2} ds$. Thus we have

$$u(t,x) = 2\nu \sum_{k=-\infty}^{\infty} b_k(t)\varphi\left(x + \frac{k\pi}{\nu}\right) .$$

The interested reader is refered to [18,19] for more details. For other essential results concerning the differential operators of infinite order and their applications, peruse [20-24].

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